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Coevolutionary dynamics of networks and games under birth-death and birth mechanisms

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Abstract. There is much interest in studying evolutionary games in structured populations, in order to understand how cooperation emerges in communities of egoistic agents. In this paper, we consider a new mechanism for cooperation to survive on networks. Agents are designed to reproduce offspring in proportion to their fitness, i.e., the aggregate payoffs they collected in the previous Prisoner's Dilemma game with neighbours. The population then evolves either by an individual giving birth to an offspring that takes over a random neighbour (birth-death process dynamics as the competition for resources already occupied by others) or by an individual constructing one new site from unexploited resources for its offspring (birth process dynamics as the competition for the unexploited resources which induces the extension of system). The underlying interaction network thus evolves and expands simultaneously with the population dynamics. The birth process dynamics is proved to be one new route that favours cooperators, under which cooperators can successfully resist the invasion of defectors in spite of large cost. Furthermore, under this "birth-death & birth" mechanism, the resulting network has a scale-free degree distribution, a small-world property, and hierarchical topology.

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1 Introduction

Much attention has been paid to the statistical physics of complex systems with theoretical game interactions recently. One example is evolutionary game theory on complex networks, which has been the standard framework to address one of the most fascinating challenges, how cooperation may survive in communities of egoistic agents. Most recent studies in this area have considered a static underlying network defining the possible competitive encounters [1–6]. In these works, the effect of network structures on the evolution and stability of cooperation has been studied. Complex networks composed of a large set of interconnected vertices of various kinds are ubiquitous in nature and society. Examples include the Internet, the World Wide Web, coauthorship and citation networks of scientists, cellular and metabolic networks, etc. Many properties of these complex networks have been reported, such as, scale-free degree distribution [8,9], small-world property [7], hierarchical topology [10,11], and so on. Recently there is an increasing interest to understand the emergence of complex network structures in diverse natural systems.

As a natural extension of those aforementioned studies, an even more intriguing task is to understand how game dynamics influences network topology. Zimmermann et al. firstly discussed this question by showing a network self-organized into a steady state with highly connected cooperators [12]. Also, under the consideration that theoretical game models on evolving networks may provide a fresh view on the emergence of complex networks in systems where strategic interactions occur, there have been several recent studies of coevolutionary games on networks [14,15,13]. However, these studies only discussed the coevolution of the network with a fixed size. In contrast, it is known that most real-life networks are open and they form by the continuous addition of new vertices to the system, thus the network size N increases throughout the lifetime of the network [8,9]. That is to say, there exist new resources that can be exploited for the maintenance and expansion of systems. Inspired by this point, we suggest that the study of a growing interaction network coevolving with game dynamics is valuable.

This paper addresses the coevolutionary dynamics of growing interaction networks and the Prisoner's Dilemma

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(PD) game. We consider the multi-agent system evolving on networks under natural selection. Each agent in the system competes for the resources by reproduction. In evolutionary biology, reproduction can be genetic or cultural [16], and the agent who does well will have a higher chance to reproduce or to be imitated by others. In our model, agents play the PD game with their local neighbours, and collect aggregate payoffs. They will be selected for reproduction in proportion to fitness, a measure of the agent's performance which corresponds to the aggregate payoff. Considering agents' competition not only for the resources already occupied by others, but also for the new resources unexploited, reproduction processes are designed to take place either via the birth-death (BD) process or birth (B) process dynamics. The probability of which process to choose is controlled by a free parameter p. Here the BD process dynamics stands for agents' competition for the already occupied resources, that is an individual giving birth to an offspring that takes over a random neighbour. The B process dynamics depicts the exploitation of new resources, that is an individual constructing one unexploited new site for its offspring rather than invading any neighbours. This process induces the expansion of the system and the underlying interaction network. It has been proved that, the natural selection of the BD process can never favour cooperators [16]. However, we show that the exploitation of new resource, i.e., the B process dynamics changes the fate of cooperators. Cooperators can resist the invasion of defectors under the natural selection of BD & B processes. Moreover, the underlying interaction network is characterized by interesting topological properties.

2 The model

In the PD game there are two strategists, cooperator and defector. A cooperator (C) is someone who pays a cost c for every partner, and the partner receives a benefit $b > c$. A defector (D) pays no cost and does not distribute any benefits to anyone. The payoff matrix can be written as:

$$\begin{array}{c} \quad C \ \ D \\ \begin{array}{c} C \\ D \end{array} \begin{pmatrix} b-c & -c \\ b & 0 \end{pmatrix}. \end{array} \qquad (1)$$

Here, we set $b = 1$ for simplification without loss of generality. It is easy to see from the payoff matrix that defectors perform better irrespective of the opponent's strategy. In an unstructured population, where all individuals are equally likely to interact with each other, defectors have a higher average payoff than unconditional cooperators. Therefore, natural selection increases the relative abundance of defectors and drives cooperators to extinction. These evolutionary dynamics hold for the deterministic setting of the replicator equation [17,18] and for stochastic game dynamics of finite populations [19].

In our model, the agents playing the evolutionary game occupy the vertices of the underlying network which evolves with game dynamics. The edges of the network denote links between individuals in terms of game dynamical interaction and biological reproduction. A cooperator that has k neighbours of which l are cooperators will get payoff $l - ck$. A defector connected to k' cooperators will get payoff k', for it pays no cost to neighbours but receives the benefit by exploiting neighbouring cooperators. The fitness of an agent is given by $f = 1 - w + wP$, where P denotes the agent's aggregate payoff throughout one generation and w measures the intensity of selection. Following previous work [16], we just consider the weak selection case with $w = 0.01$.

The system evolves according to the following rule: In each generation, agents play the PD game with neighbours and collect aggregate payoffs. Then, one agent i in the system is selected with probability $\Pi(f_i) = f_i/F$ for reproduction, with f_i the fitness of agent i, and $F = \Sigma_j f_j$ the total fitness of the system. This selected agent takes the BD process with probability p, or the B process instead with probability $1 - p$. In the BD process, one randomly chosen neighbour of this selected agent is replaced by its offspring. In the B process this agent constructs one unexploited new site for its offspring without invading any of its neighbours. That is to say, one new vertex occupied by the offspring will be inserted into the environment of the parent (i.e. the selected agent) with m new edges, among which $m - 1$ edges are linked to randomly chosen neighbours of the parent and one edge is linked to the parent itself. Here, multiple edges or self-edges are not allowed. The network size will increase by one if the B process takes place. After t time steps this algorithm produces a network with approximately $N_t = N_0 + pt$ vertices and $\frac{N_0}{2}k_0 + mpt$ edges. Here, N_0 and k_0 respectively are the size and average degree of the initial network.

Simulations are carried out from an initial network which is a one-dimensional lattice with periodic boundary conditions and coordination number $z = 4$ [20]. The initial network size N_0 is set as 60. In the birth process, the number of edges one new vertex brings in is $m = 4$, which is equal to the coordination number z. In the initial state, cooperators and defectors are uniformly distributed among all the sites.

3 Results and analysis

The B process supports the intuitive result that agents with high fitness will have a higher chance to exploit new resources, opening a new route to the evolution of cooperation: Those cooperators whose fitness are comparatively high will have more chance to generate offspring, and at the same time let their offsprings possess their good environment. However, the defectors, who are competing for the reproduction chance against cooperators, will also take advantage of the B process; and moreover, the development of defectors will deteriorate cooperators' environment further. Therefore, it remains an open problem who — defectors or cooperators — will be able to profit from the B process, and to which extent.

3.1 Birth-death process

Firstly, let us consider the effect of the pure BD process. At each time step, one agent i is selected for reproduction with the foresaid probability $\Pi(f_i)$ proportional to fitness, and the offspring replaces a randomly chosen neighbour of i. The state of agent i is denoted as $s(i)$, which equals 0 or 1 when it is a defector or a cooperator, respectively. Then, the probability for the offspring to replace those neighbours with states different from $s(i)$ is,

$$Prob(i, s(i)) = \sum_j \frac{A_{ij}}{k_i}\{s(i)[1 - s(j)] + s(j)[1 - s(i)]\}. \quad (2)$$

Here, A_{ij} is the adjacency matrix whose elements equal to 1 if ij are connected and zero otherwise ($j = 1, \dots, N$). Thus, we know that, for an elemental update of BD process, the number of agents with state S (D or C), denoted as N_S, will increase by one with probability I_{N_S+1}, and decrease by one with probability I_{N_S-1}, where

$$I_{N_S+1} = \sum_{i=1, s(i)=S}^{N} \Pi(f_i) Prob(i, s(i)) \quad (3)$$

$$I_{N_S-1} = \sum_{i=1, s(i)=S'}^{N} \Pi(f_i) Prob(i, s(i)). \quad (4)$$

Here, state S' represents the opposite state of S. We can see that, the BD process of the agent surrounded by the identical-strategy neighbours would make no sense (the case with $A_{ij} = 1$ and $s(i) = s(j)$ in Eq. (2)), and only that of the agents right at the boundary between cooperators and defectors matter (the case with $A_{ij} = 1$ and $s(i) \neq s(j)$ in Eq. (2)). However, cooperators at the boundary are always at a disadvantage for they are exploited by defectors. The pair-approximation and diffusion-approximation calculations given in reference [16] have proved that the BD selection can never favour cooperators, and defectors always dominate the whole system.

3.2 Birth-death & birth process

However, cooperators can survive when there exist unexploited resources, which ensure the B process dynamics of the high-fitness agents. Figure 1 plots the probability densities of cooperators ρ_C as a function of network size during evolution of the network under birth-death & birth (BD & B) selection (take the case with $c = 0.8$ and $p = 0.9$ as an example). It can be easily seen from the figure that after a long time evolution, the frequency of cooperator ρ_C finally evolves to nonzero plateau values denoted as ρ_∞, in spite of the high cost c and the rare occurrence of the B process (i.e., large p). This remarkable result shows clearly that cooperators benefit more from the B process than defectors.

We argue that this can be understand as follows: cooperator's offspring will not only take advantage of the
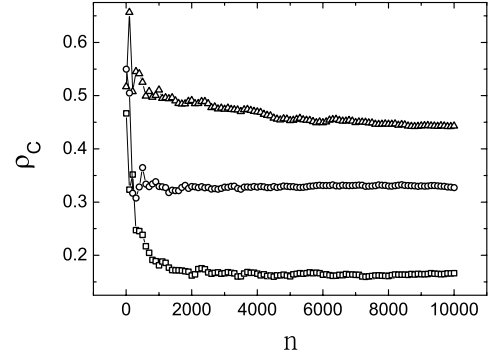
Fig. 1. The probability density of cooperators ρ_C at the time when the nth new vertex is inserted, until the system size reaches $N = 10^4$, with a high cost $c = 0.8$ and $p = 0.9$. The three different symbols correspond respectively to the evolution of the initial network with three different initial distributions of C and D.

good environment of its parent, but also of the parent's protection. In return, as a neighbouring cooperator, the offspring will also protect its parent. This mutual protection may increase cooperators' fitness, while defectors do not have this good nature.

Now, let us consider the B process of a cooperator C_1 and a defector D_1. For C_1 with k neighbours and D_1 with k' neighbours, their payoffs will be,

$$P_{C_1} = \eta k - ck \quad (5)$$
$$P_{D_1} = \eta' k'. \quad (6)$$

Here, η (or η') is the fraction of cooperators among the neighbours of C_1 (or D_1). Then, one get the fitness $f_{C_1} = 1 - w + wP_{C_1}$, and $f_{D_1} = 1 - w + wP_{D_1}$. In the B process, their new offsprings C_2 and D_2 are inserted into their environments, and new environments are established as a consequence. The payoffs of the parents and the offsprings turn out to be,

$$P'_{C_1} = \eta k - ck + 1 - c = P_{C_1} + 1 - c \quad (7)$$
$$P'_{D_1} = \eta' k' = P_{D_1} \quad (8)$$
$$P_{C_2} = 1 + \eta(m - 1) - cm \quad (9)$$
$$P_{D_2} = \eta'(m - 1). \quad (10)$$

Here, P'_{C_1} and P'_{D_1} respectively are the new payoffs of C_1 and D_1 after their B processes. From equations (7) and (8) one can find that, reproducing an offspring is beneficial to the cooperator parent, for the payoff of the parent C_1 increases by $\Delta P_{C_1} = 1 - c$ compared to the previous state. On the other hand, reproduction does not favour a defector's parent (see Eq. (8)).

For the purpose of comparison, let us consider the B processes of C_1 and D_1 with the same fitness and the same number of neighbours, which corresponds to $P_{C_1} = P_{D_1}$ and $k = k'$ (Fig. 2 illustrates the special case with $k = 5$ and $m = 4$). From equations (5) and (6), we get

$$\eta' = \eta - c. \quad (11)$$
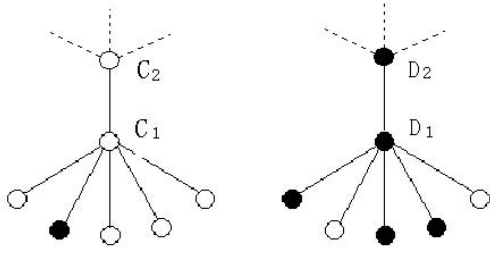
Fig. 2. The black cycles are defectors, and the white cycles are cooperators. As is shown, the payoffs of C_1 and D_1 are both equal to 2 before the B process (with $b = 1$ and $c = 0.4$). The C_2 and D_2 respectively are the offsprings of C_1 and D_1.

Then, one can easily obtain the following relations from equations (7) to (11),

$$P'_{C_1} = P'_{D_1} + 1 - c \tag{12}$$
$$P_{C_2} = P_{D_2} + 1 - c. \tag{13}$$

Interestingly, in the case that the parents' fitness and degree are the same, after the B process, both the parent and the offspring of cooperators can perform better than those of defectors. It can be imagined that, in the next generation the better performance of cooperators will bring them more chance to be selected for reproduction. Thus, cooperators obviously take advantage of B process more than defectors.

The above analysis of the two process dynamics and the sketch map of the simple case exhibits a clear picture as to why cooperators can survive during the network evolution under BD & B selection. The BD process dynamics (with probability p) favours defectors, while the B process dynamics (with probability $1 - p$) results in the prosperity of cooperators. Then the system under BD & B selection evolves to a stationary regime with the coexistence of cooperators and defectors.

As shown in Figure 1, the final plateau value ρ_∞ is sensitive to the initial distribution of C and D. We thus study the statistical properties of simulation results. Results from 2000 different initial distributions of C and D on the initial network are considered. The histogram of ρ_∞ denoted as $\xi(\rho_\infty)$ are plotted in Figure 3. We can see that the values ρ_∞ depend on the dynamical parameters p and c. For example, in the case with $p = 0.95$, the asymmetry distribution biased to the small values appears. This bias increases with the increment of cost. Figure 4a shows the average value $\langle \rho_\infty \rangle$ over the 2000 simulation results as a function of cost c. It is seen that the increment of the cost will reduce the density of cooperators irrespective of the value of p. Also, one can notice that the smaller value of p corresponds to the smaller slope of the plots. That is to say, in the coevolutionary process if more B processes take place (smaller p), the fraction of cooperator in the system will decrease slower with the increase of the cost. Thus, we can say that, the effect of the B process counteracts the harmful influence of the cost. In addition, from the figure we know that even for the case with very high cost and rare birth processes (such as $c = 0.9$ and $p = 0.95$ case), cooperators still exist with a comparatively high density.
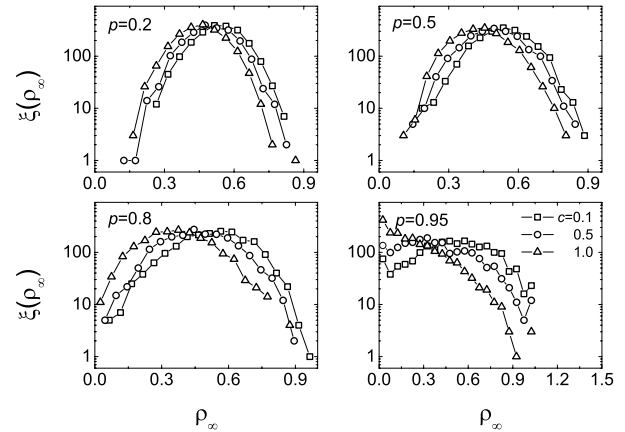
Fig. 3. The log-normal plots of the histogram of the values ρ_∞, denoted as $\xi(\rho_\infty)$, from the simulation results of 2000 different initial distributions of C and D. The four figures are the results from cases with $p = 0.2$, 0.5, 0.8, and 0.95, respectively. The system size $N = 10^4$, and the cost $c = 0.1$, 0.5, and 1.0.
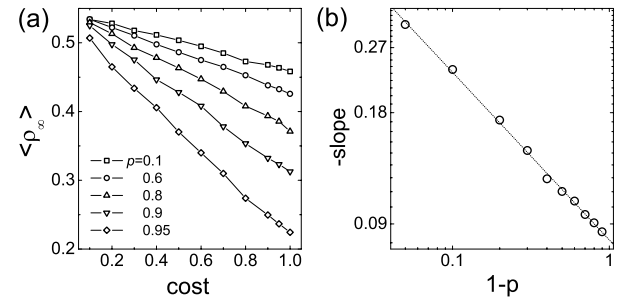

Fig. 4. (a) The average value of the final plateau values ρ_∞ from 2000 different initial distributions of C and D as a function of cost, with $p = 0.1$, 0.6, 0.8, 0.9, and 0.95, respectively. (b) The slopes of the lines in (a) as a function of $1 - p$. The scaling behavior of the log-log plots implies the relation, $slope \sim -(1 - p)^{-\alpha}$. The dot line is power-law regression fit, which gives an estimate of power-law exponent $\alpha \simeq 0.454$.

Furthermore, in Figure 4b we present the fitted slopes of those plots in (a) as a function of $1 - p$. It is noticeable that the fitted slope scales as $-(1 - p)^{-\alpha}$ with $\alpha \simeq 0.454$.

3.3 The resulting networks

We next analyze some properties of the resulting network from the evolutions under BD & B selection. We first calculate the cumulative degree distributions (see Fig. 5). As many real-life networks, the resulting networks obtained from the cases with different dynamical parameters have identically scale-free degree distribution, $P(k) \sim k^{-\gamma}$. Table 1 lists the estimated power-law exponents γ for each of the distributions of Figure 5, calculated using the maximum likelihood method with the bottom of the lowest bin at $K_{min} = m$ [21]. It is interesting that these networks have similar power-law exponents as the BA network [8,9]. We can explain this point as the result of their two common ingredients, "growth" and "preferential attachment" (PA). The reproduction proportional to fitness
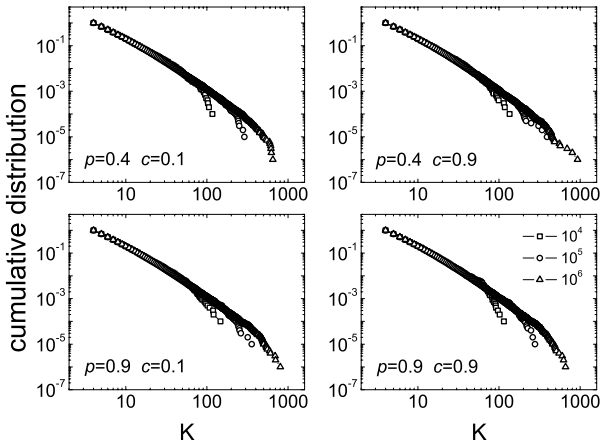
Fig. 5. Cumulative degree distributions of the resulting networks with size $N = 10^4$, 10^5, and 10^6 at different values of p and c.

Table 1. The values of the power-law exponents γ of resulting networks with size $N = 10^4$, 10^5, and 10^6. Numbers in parentheses give the standard error on the trailing figures.

p	0.4		0.9	
c	0.1	0.9	0.1	0.9
10^4	3.05(2)	3.04(2)	3.04(2)	3.03(2)
10^5	3.056(7)	3.047(6)	3.058(6)	3.050(6)
10^6	3.056(2)	3.050(2)	3.057(2)	3.050(2)
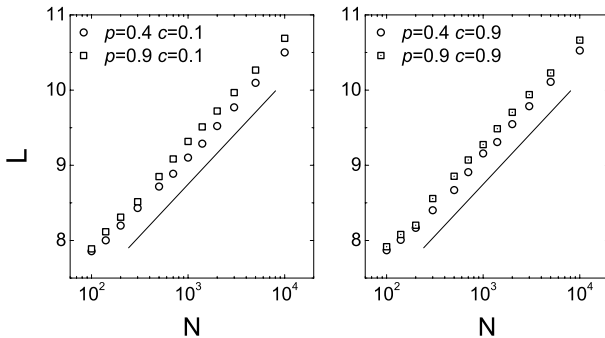
Fig. 6. The log-normal plots of the average path length L as a function of N, with different values of p and c.

in our model and the degree-PA mechanism proposed by the BA model may induce structural similarity by reason that the higher degree agents in our model have more probability to gain larger profits in the game (i.e. to have larger fitness). In spite of the similarity of degree distribution of the resulting networks and BA networks, further analytical results will show that they are essentially different in some aspects.

The average path length and the clustering coefficient of the evolving networks are plotted in Figures 6 and 7, respectively. From the straight line form in the log-normal figure of the average path length L, one get that L scales logarithmically with the number of vertices, i.e., $L \sim \ln N$, which is coincident with the result in reference [22]. On the other hand, we can see from Figure 7 that, for different
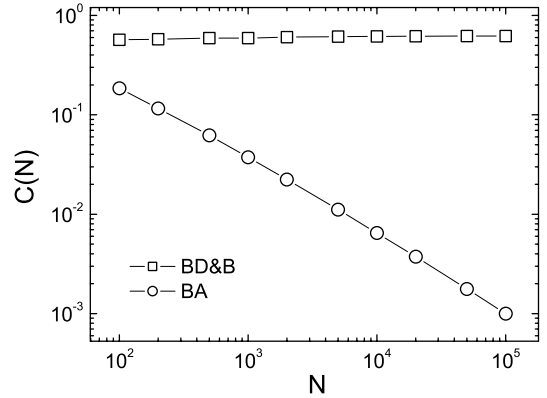

Fig. 7. The dependence of the clustering coefficient C on the size of the network N. While for the resulting network from BD & B selection ($\square$) C is independent of N at large size, for the scale-free BA model ($\bigcirc$) $C(N)$ decreases rapidly. Here we take the case with $p = 0.4$ and $c = 0.5$ as an example.

values of p and c, the clustering coefficient of the resulting network increases with the network size N from the initial value 0.5 to a larger value (about 0.62). When the network size exceeds 10^4 the clustering coefficient becomes independent of the size. On the contrary, the clustering coefficient of BA networks decreases with the network size N (see Fig. 7). As we know, "small-world networks" are characterized by a high degree of clustering and a small average path length [23]. Thus, from the above analysis, we can say the resulting networks from the evolution under BD & B selection are small-world networks.

Besides the scale-free degree distribution and the small-world property, hierarchical topology is also an important property of many realistic networks [10]. It was mentioned in reference [11] that the clustering coefficient of a vertex with k links in their deterministic scale-free networks follows the scaling law

$$C(k) \sim k^{-1}, \tag{14}$$

which characterizes the intrinsic hierarchy of networks. We also study the hierarchy of the resulting networks by this quantitative manner. After averaging over the clustering coefficients of all the vertices for a given degree k, the results $C(k)$ as a function of vertex degree are obtained. The average results over 1000 realizations are depicted in Figure 8, where the log-log plots of the $C(k)$ curves with exponents approximately equal to 1 demonstrate that they follow equation (14). Figure 8a shows the average results with $p = 0.4$ and $c = 0.5$ at different network sizes. From the overlapping of the results at the small size $N = 500$ and the larger sizes, we can see that the scaling law emerges early in the evolution under BD & B selection. Furthermore, from the results with different values of dynamical parameters plotted in Figure 8b we know that the hierarchical topology is a general property in our model.
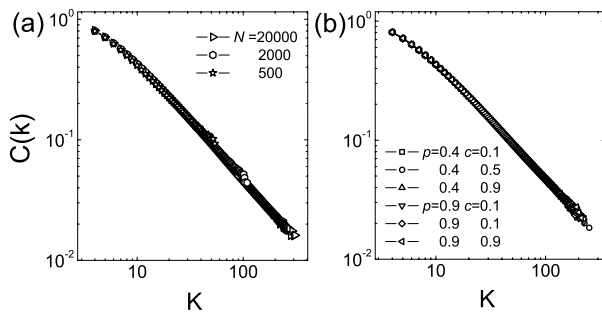
Fig. 8. (a) The clustering coefficient C averaged over the results of given degree k, with $p = 0.4$ and $c = 0.5$ as an example. The three different symbols correspond to the results at different moments of the same evolution with system sizes $N = 500$, 2000 and 20000. (b) The correlation of clustering coefficient C and degree k with different values of p and c. The results shown here are averaged over 1000 realizations of the networks.

4 Conclusion

In summary, we have studied the coevolutionary Prisoner's Dilemma game on an evolving network where each agent competes for the chance to reproduce according to the birth-death & birth selection rule. We find that the birth process dynamics, in which agents make use of the unexploited resources for reproduction, favours cooperators. We thus argue that, the availability of unexploited resources might be a new route for the emergence of cooperation in real-world systems. On the other hand, the underlying interaction network in our model evolves driven by game dynamics, and the resulting network is found to have a scale-free degree distribution, a small-world property, and hierarchical topology. Thus, our work also provides a mechanism for the evolution of complex networks possibly relevant for real-world systems.

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